

Recursion relations, helicity amplitudes and dimensional regularization

Callum Quigley

*Department of Mathematics, University of Toronto
Toronto, Ontario M5S 2E4, Canada
E-mail: cquigley@math.utoronto.ca*

Moshe Rozali

*Department of Physics and Astronomy, University of British Columbia
Vancouver, British Columbia V6T 1Z1, Canada
E-mail: rozali@phas.ubc.ca*

ABSTRACT: Using the method of on-shell recursion relations we compute tree level amplitudes including D-dimensional scalars and fermions. These tree level amplitudes are needed for calculations of one-loop amplitudes in QCD involving external quarks and gluons.

KEYWORDS: Renormalization Regularization and Renormalons, Hadronic Colliders.

JHEP03(2006)004

Contents

1. Introduction and summary	1
2. Preliminaries	3
2.1 Notations	3
2.2 One-loop amplitudes with quarks and gluons	4
2.3 D-dimensional fields	5
2.4 Recursion relations	7
2.5 Primitive vertices	9
2.6 Checks on the amplitudes	11
3. Four point amplitudes	11
3.1 Type-A amplitudes	12
3.2 Type-B amplitudes	13
3.3 Type-C amplitudes	15
4. Five point amplitudes	16
4.1 Type-A amplitudes	16
4.2 Type-B amplitudes	17
4.3 Type-C amplitudes	18
A. Collinear limits	20

1. Introduction and summary

The last two years have seen the development of new and surprising techniques for performing perturbative calculations in gauge theories. Following the seminal paper by Witten [1] (drawing on earlier insights by Nair [2]), quantifying and generalizing the simplicity of some tree level amplitudes [3], the initial efforts have been focused on the relation to twistor space (for a review see [4]). These investigations resulted in a beautiful effective Feynman diagram technique [5], the so-called MHV diagrams or CSW rules, which can be applied to calculations of tree level [6], one-loop [7] amplitudes and amplitudes with external massive sources [8]. Recently it has been extended to gravity amplitudes as well [9].

Though the CSW diagrammatic method proved much more efficient in calculating amplitudes than the traditional method of Feynman diagrams, they still have the same flavor, and it turns out there exists an even more efficient method, that of the on-shell

recursion relation. This method was suggested in [10] (see also [12]), and was proven by [11]. It seems applicable to calculations of tree level amplitude (or more generally rational functions) in a wide range of theories, and has the flavor of the analytic S-matrix theory, in that it does not make use of off-shell structures. As such it seems like a genuinely new way of performing calculations in perturbative quantum field theories.

The recursion relations are reviewed in section 2 below, together with other background material. They have been utilized to calculate tree level amplitudes in gauge theories [13], gravity [14], amplitudes including massive sources [15–17] and rational functions appearing in one-loop amplitudes [18]. In all these cases the results obtained are either new, or are a more compact form of previously calculated results. The relation to the CSW method was discussed recently in [19].

A more challenging task, and one of relevance to upcoming experiments at the LHC, is the calculation of one-loop amplitudes. The main ingredient used in calculating one loop amplitude is that of unitarity: the multi-valued part of the loop amplitude is determined by the tree level results (see for example [20, 21]). The compactness of the results for the tree level amplitudes is extremely useful when they are used as input for the calculation of one-loop amplitudes.

The method of generalized unitarity is one of the most efficient general methods of using the knowledge of tree level amplitudes in calculating the one-loop amplitudes. It has been discussed recently in [22, 23]. In particular then discussion in [22] has concentrated on cut-constructible amplitudes (in the sense of [24]). The more general amplitude has rational pieces, which can sometimes be determined separately using recursion relations [25].

However, as explained in [23], a systematic method to obtain the complete amplitude, including any rational parts, is using generalized unitarity in D-dimensions. In continuing away from four dimensions, in dimensional regularization, the rational pieces acquire cuts as well, and therefore can be constructed using generalized unitarity. The form of the one-loop amplitude thus constructed is expected to be compact, reflecting the simplicity of their building blocks, the tree level diagrams.

Motivated by this line of development, we calculate below all the tree level amplitudes needed for calculating one-loop amplitudes with up to five partons. These tree level amplitudes differ from the ones previously calculated in that some of the external legs are continued to D-dimensions. The one loop amplitudes with 5 partons were previously calculated in [26], and we can check the generalized unitarity method against those explicit results. Calculations of one loop amplitudes in QCD, with up to six partons, are in progress [27].

The outline of this paper is as follows: in section 2 we motivate the set of tree level amplitudes constructed here, as the building blocks for the aforementioned one-loop amplitudes. The method we use is the extension of the BCF recursion relations, and we explain the new issues arising when including D-dimensional scalar and fermions. In section 4 we exemplify the method by calculating the four point amplitudes in detail. Section A is devoted to calculations of the five point amplitudes and some checks on them.

2. Preliminaries

2.1 Notations

A massless momentum in four dimension can be written as a product of two (bosonic) spinors which we denote by λ_α and $\tilde{\lambda}_{\dot{\alpha}}$, so that $p_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^\mu p_\mu = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$, where $\sigma^\mu = (1, \vec{\sigma})$, and $\vec{\sigma}$ are the Pauli matrices. We also denote alternatively $\lambda_\alpha = |\lambda\rangle$ and $\tilde{\lambda}_{\dot{\alpha}} = [\lambda]$, so that $p = |\lambda\rangle[\lambda]$. In the case of several momenta, we also shorten $|\lambda_i\rangle$ and $[\lambda_i]$ to $|i\rangle$ and $[i]$ respectively.

For four dimensional fermions we denote by $u_\pm(k)$ two of the positive energy solutions of the massless Dirac equation $\not{k}u_\pm(k) = 0$ of helicities $\pm\frac{1}{2}$. These solutions are denoted $u_+(k) = |\lambda\rangle = \lambda_\alpha$ and $u_-(k) = |\lambda] = \lambda^{\dot{\alpha}}$, and they are eigenspinors of γ^5 with eigenvalues \mp respectively.

The internal products of these spinors are defined as

$$\langle ij \rangle = \bar{u}_-(k_i)u_+(k_j) \quad [ij] = \bar{u}_+(k_i)u_-(k_j) \quad (2.1)$$

which also defines the dual (bra) of each spinor (ket). With this notations $|i] = \lambda_i^{\dot{\alpha}}$ and $[i| = (\lambda_i)_\alpha$ are positive chirality spinors of opposite (\mp) helicities, whereas $|i\rangle = (\lambda_i)_\alpha$ and $\langle i| = \lambda_i^\alpha$ are negative chirality spinors of opposite (\pm) helicities (note that the internal products are then Lorentz scalars). These consist of the four positive energy solutions of the massless Dirac equation. One has the following identities

$$|i\rangle[i| = \omega_+ k\!\!\!/\quad |i][i| = \omega_- k\!\!\!/\quad (2.2)$$

where $\omega_\pm = \frac{1}{2}(1 \pm \gamma^5)$ are projections onto the positive or negative chirality subspaces.

Wave-functions for external gluons can be written in this basis as bi-spinors $\epsilon_{\alpha\dot{\alpha}}^\pm$

$$\epsilon^+(k) = \frac{|q\rangle[k|}{\sqrt{2}\langle qk \rangle} \quad \epsilon^-(k) = -\frac{|k\rangle[q|}{\sqrt{2}[qk]}, \quad (2.3)$$

where q is an arbitrary reference momentum, changing it amounts to a gauge transformation on the gluon polarization vector. Similar expression hold for the conjugate part of the polarization (ϵ^\pm) $^{\dot{\alpha}\alpha}$. Working with $\epsilon_{\alpha\dot{\alpha}}^\pm$ amounts to concentrating on a negative helicity Weyl spinor, which does not mix with the positive helicity one in a purely massless theory.

In performing helicity amplitude calculations, it is customary to include the wave-functions for internal fermions or gluons in the interaction vertices rather than in the propagator. Therefore the propagator is always the scalar propagator $\frac{1}{p^2}$, and numerators which usually accompany fermion propagators come about from the two interaction vertices connected by the given propagator.¹

For other conventions and notation used in helicity amplitude calculations, we refer the reader to the review [21].

¹For fermions in a complex representation of the gauge group there is a distinction between fermion and anti-fermion. As the formalism treats them uniformly, one has to add by hand a minus sign for each anti-fermion line.

2.2 One-loop amplitudes with quarks and gluons

The tree level amplitudes calculated here are to be used as basic building blocks in evaluating one-loop diagrams in pure QCD, with quark-antiquark pair, and a number of external gluons. We briefly describe here this calculation, using generalized unitarity in $D = 4 - 2\epsilon$ dimension, leaving the details to future work [27]. This serves as a motivation for the particular set of tree level amplitudes we evaluate in this paper.

We will concentrate on color-ordered (or partial) amplitudes, stripping the color indices off the external legs. We will therefore not distinguish between various matter representations.² For the purposes of QCD, the fermions are in the fundamental representation, and discussion of the color ordered amplitudes in this context can be found e.g in [28].

Another standard tool is the supersymmetric decomposition of amplitudes. The one-loop amplitudes are easier to calculate if a complete supersymmetric multiplet runs in the loop, as loop amplitudes are cut constructible in those cases [24]. This allows us to trade some combinations of particles running in the loop for others. For our case, it is sufficient then to calculate loop amplitudes with scalars and fermions (but no gluons) appearing in the loop, and (as mentioned above) external quarks and gluons.

The one loop amplitudes can be calculated if we know their singularities in $D = 4 - 2\epsilon$ dimensions. This requires knowledge of tree amplitudes with two external legs continues to D dimensions. Using four dimensional helicity regularization (i.e, where the momenta but not the polarizations are continued to D dimensions), the D dimensional external momenta can be thought of as 4 dimensional massive momenta [29]. Inspection of the quadruple and triple cuts of the aforementioned one-loop amplitudes leads to the following 4 sets of tree level amplitudes (see figure 1), which will be calculated below:

- Amplitudes including 2 massive (or massless in $D = 4 - 2\epsilon$ dimensions) scalars, and some number of gluons. These were already calculated in the first reference in [15].
- Amplitudes including 2 massive fermions (which we denote by λ), and some external gluons. We will label these type-A amplitudes.
- Amplitudes including 2 massive scalars, and massless gluons, accompanied by an external quark-antiquark pair. We will label these type-B amplitudes.
- Amplitudes including a massive scalar and massive fermions, with external massless fermion and a few gluons. Those amplitudes will be labeled type-C.

We note that all amplitudes involve two massive legs which are adjacent.

In an effort to use a uniform notation, we will denote the momenta of external³ fermions by k , external scalars by l , and external gluon by p . In addition, as explained in the next subsection we need to distinguish between four dimensional momenta (denoted by small letters) and D-dimensional ones (denoted by capital letters).

²For example, we will not assume the fermions to be adjacent, as could be assumed for quarks, to include the possibility of external gluinos.

³We will not use uniform notation for intermediate state momenta that are eliminated from the final result anyhow.

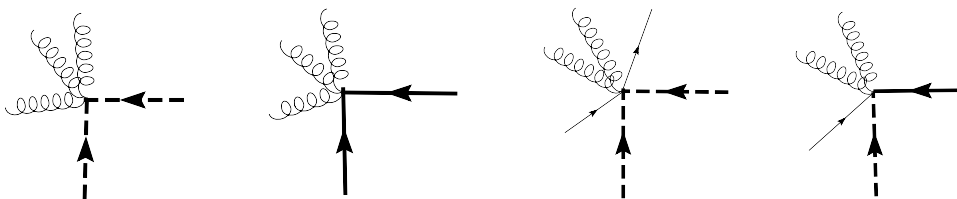


Figure 1: The relevant tree diagrams, with varying number of gluons, are ordered from left to right. The massive legs are denoted in boldface lines, solid lines denote fermions, dashed lines scalars, and wiggly lines gluons.

2.3 D-dimensional fields

We find that in order to use D-dimensional unitarity, we have to use helicity methods in calculating tree level amplitudes in D-dimensions. In the FDH regularization scheme, the momentum is continued to D-dimension. Every D-dimensional momentum P can be decomposed as $P = p + \mu$, where p is the four dimensional component, and μ is a component in a formal (-2ϵ) -dimensional orthogonal space.⁴ Working in mostly minus signature, $P^2 = p^2 - \mu^2$, so on shell massless momentum ($P^2 = 0$) is equivalent to four-dimensional massive momentum $p^2 = \mu^2$. Therefore for scalars, working away from 4 dimensions is equivalent to adding mass to the scalar field.

Each loop momentum integration can be decomposed as $\frac{d^D P}{(2\pi)^D} = \frac{d^4 p}{(2\pi)^4} \frac{d^{(-2\epsilon)} \mu}{(2\pi)^4}$, so the mass μ is always integrated over, and the ϵ dependence of the amplitude is generated from the μ dependence of the integrand. Similarly, each vertex is accompanied by a delta function imposing momentum conservation. As all momenta are now D-dimensional, those are also D-dimensional delta functions. It is therefore necessary (in general) to regard the momenta of internal lines as being D-dimensional as well; for tree level amplitudes they are just given as linear combinations of external momenta, given by imposing all the momentum conservation constraints.

For internal fermionic lines one always has to sum over the intermediate spinor wavefunctions, so choice of basis is not necessary. We will therefore use the notation $|P\rangle, \{P|$ to refer collectively to these wavefunctions. The sum over the intermediate wavefunction is performed using the identity

$$|P\rangle\{P| = \not{P} \tag{2.4}$$

Note that now \not{P} has one component (\not{p}) that preserves helicity and one that flips it ($\not{\mu}$). similarly, in D-dimensions the components \not{p} and $\not{\mu}$ behave differently with respect to chirality, $\{\not{p}, \gamma^5\} = 0$ whereas $[\not{\mu}, \gamma^5] = 0$.

To mimic the four dimensional helicity methods, we want to utilize helicity-like states for external fermions, even when they are D-dimensional. As we keep γ^5 four dimensional, we can still use chiral basis which we denote as before by $|P\rangle = \omega_+ |P\rangle$ and $|P] = \omega_- |P]$ (and similarly the conjugates $\langle P|$ and $[P|$). However the states of definite helicity, $|P\rangle$ and $[P|$ (or similarly $|P]$ and $\langle P|$) now mix with each other. Indeed, the basis vectors $|P\rangle$ and

⁴We will use the capital letters for D-dimensional momenta, and small letters for their four dimensional components. The (-2ϵ) component of all D-dimensional momenta is always $\pm\mu$.

$|P\rangle$ do not individually satisfy the massless Dirac equation in D-dimensions. That equation written in terms of these Weyl fermions is,

$$\not{p}|p\rangle + \not{\mu}|p\rangle = 0 \quad \not{p}|p\rangle + \not{\mu}|p\rangle = 0 \quad (2.5)$$

which is consistent with the mass-shell condition $p^2 = \mu^2$.

Nevertheless one can assemble the physical amplitudes (with external wave functions $|P\rangle = |P\rangle + |P\rangle$) from the ones calculated here, as helicity violation is limited to insertions of $\not{\mu}$ in fermion lines. In the course of using the recursion relations we demonstrate this process, which is also relevant for the calculation of one-loop amplitudes using generalized unitarity [27]. Whenever one encounters an intermediate D-dimensional fermion, one can write the numerator of the propagator as

$$\not{P} = |P\rangle\langle P| = (|P\rangle + |P\rangle)(\langle P| + \langle P|) \quad (2.6)$$

and make use of the partial amplitudes with helicity states, the ones we calculate here.

Note however that the propagator has both helicity preserving and helicity flipping parts. The helicity preserving parts are the usual propagators, usually drawn as connecting \pm states to \mp states,

$$|P\rangle\langle P| = \omega_+ \not{p} \quad |P\rangle\langle P| = \omega_- \not{p} \quad (2.7)$$

whereas the helicity flipping parts are new, and are the part of the propagator that connects \pm states to \pm at the other end of the propagator. They arise from the identities

$$|P\rangle\langle P| = \omega_- \not{\mu} \quad |P\rangle\langle P| = \omega_+ \not{\mu}. \quad (2.8)$$

The main advantage of using chiral external states is that one can use the simple expressions for the gluon polarizations,

$$\begin{aligned} \epsilon^+(k) &= \frac{1}{\sqrt{2}\langle qk\rangle} (|q\rangle\langle k| + |k\rangle\langle q|) = (\epsilon^+)_{\alpha\dot{\alpha}} + (\epsilon^+)^{\dot{\alpha}\alpha} \\ \epsilon^-(k) &= -\frac{1}{\sqrt{2}[qk]} (|q\rangle\langle k| + |k\rangle\langle q|) = (\epsilon^-)_{\alpha\dot{\alpha}} + (\epsilon^-)^{\dot{\alpha}\alpha}. \end{aligned} \quad (2.9)$$

This polarization contracts into the fermionic states that accompany the gluon in an interaction vertex. In case at least one of these states is a Weyl fermion, one of the terms in the polarization vanishes (which one depends on the chirality of the fermionic state). Note that in this case the chirality of the other fermion is determined, even if it is a D-dimensional (and thus Dirac) fermion.

To summarize the step of the calculations, we will be using the BCF recursion relation to find an expression for helicity amplitude which is valid in general dimension D (in the FDH scheme). Subsequent manipulations will depend on which external (and internal) legs are taken to be D-dimensional, those include translating the helicity states into momenta, using mass shell conditions and trace identities to simplify the results. We will be very explicit in calculations of the four point amplitudes in section 3, to demonstrate the issues involved, and will be less detailed in deriving the five point amplitudes.

2.4 Recursion relations

To evaluate the tree level amplitudes, we will use the recursion relations first discussed in [10] and proven in [11]. The proof utilizes the analytic properties of rational functions, and can be then generalized to massive particles [29], and to purely rational loop amplitudes [7], or to calculating the rational parts of loop amplitudes [18]. Here we slightly generalize it for the case of D-dimensional fields. We briefly review the method as needed for our purposes, highlighting the slight differences arising in our case. We refer the reader to a more detailed discussion in the original papers.

We will be discussing tree level amplitudes $\mathcal{A}_n(p_1, \dots, p_n)$ of n external on-shell particles, $n - 2$ of which are massless in 4 dimensions, or two are taken to be massless in $D = 4 - 2\epsilon$ dimensions (or equivalently massive in 4 dimensions). The recursion relations depend on choosing two of the external momenta (labeled i, j) and "marking" them.

Now, define a function $\mathcal{A}(z)$ to be the amplitude evaluated at the shifted momenta⁵ $\hat{p}_i = p_i + z\eta$ and $\hat{p}_j = p_j - z\eta$, where η is a null vector orthogonal to both p_i, p_j . This ensures the same mass-shell condition applies to the shifted momenta. The shifted momenta are now null and complex. We will always choose at least one of the marked momenta to be massless in 4 dimensions, as this simplifies the analysis, and is sufficient for our purposes.⁶ In case both marked momenta are null, we can write them as product of spinors: $p_i = \lambda_i \tilde{\lambda}_i$ and $p_j = \lambda_j \tilde{\lambda}_j$, then $\eta = \lambda_j \tilde{\lambda}_i$. In this case the shift amounts to shifting the spinors $\lambda_i \rightarrow \lambda_i + z\lambda_j$ and $\tilde{\lambda}_j \rightarrow \tilde{\lambda}_j - z\tilde{\lambda}_i$, leaving $\tilde{\lambda}_i$ and λ_j intact. In case that p_i is massive and $p_j = |j\rangle[j]$ is massless, we have $\eta = [j|p_i|j]$ (and similarly for $i \leftrightarrow j$).

One then divides the n external momenta to two cyclically ordered groups, which are labeled $L = \{p_r, \dots, p_i, \dots, p_s\}, R = \{p_{s+1}, \dots, p_j, \dots, p_{r-1}\}$. As is indicated the groupings is such that $p_i \in L$ and $p_j \in R$, and we will sum over all such groupings. We denote by $p = p_r + \dots + p_s$, the momentum flowing in the channel between the L, R groups of momenta,⁷ and $\hat{p} = p + z\eta$ is the shifted intermediate momentum. The shift variable z is chosen to impose the appropriate mass-shell condition on the shifted intermediate momentum \hat{p} . For uniformity of notation, we impose the same mass-shell condition imposed on external legs: $\hat{p}^2 = 0$ for purely four dimensional momenta, and $\hat{p}^2 = \mu^2$ for components of D-dimensional momentum \hat{P} .

The BCF recursion relation is then

$$\mathcal{A}_n(p_1, \dots, p_n) = \sum_{L,R} \sum_h A_L(p_r, \dots, \hat{p}_i, \dots, p_s, -\hat{p}^h) \frac{1}{p^2} A_R(\hat{p}^{-h}, p_{s+1}, \dots, \hat{p}_j, \dots, p_{r-1}). \quad (2.10)$$

The first sum is over all possible groupings of external momenta, as described above. The second sum is over all possible intermediate states (depending on the matter content of

⁵Note that the momentum conservation constraint is unaltered by the shift, so $\mathcal{A}(z)$ can be calculated using perturbation theory.

⁶For simplicity we take η to be purely four dimensional null momentum, even when D-dimensional momenta are involved.

⁷To conform with the notation in this paper, we use p for four dimensional intermediate momentum, and P for a D-dimensional such momentum.

the theory), and their helicities⁸ h . The amplitudes A_L, A_R depend on shifted momenta as indicated, and are momentum conserving on-shell amplitudes, albeit with complex momentum; they include an additional external leg with momentum $\pm\hat{p}$ and the appropriate helicity $\pm h$. Note also that the momentum p appearing in the propagator is unshifted.

The validity of the recursion relations depends crucially on one technical assumption, that of the vanishing of the function $\mathcal{A}(z)$ as $z \rightarrow \infty$. This depends on the helicities of the marked momenta i, j . In general, the helicities (h_i, h_j) must be chosen so that $h_i \geq h_j$, with additional constraints when choosing quarks. In particular, for two gluons we cannot choose $(-, +)$ [10]; when choosing a gluon and a scalar, positive helicity gluons must be particle i , while negative helicity gluons must be in the position j [15]. When “marking” a quark, we cannot also choose an adjacent quark nor an adjacent scalar, and for an adjacent gluon-quark pair, we have similar rule as above: positive and negative helicity gluons must be chosen as i and j , respectively [16].

We would now like to resolve the issue of choosing an adjacent gluon and quark (in that order) with helicities $(+, +)$. While [13] claims that this choice is invalid, we will argue that it is in fact allowed, as stated in [16]. We will consider $\mathcal{A}(z)$ as a sum of Feynman diagrams where the momenta of the gluon i and adjacent quark j depend on z , and we will write $\hat{p}_i = p(z)$ and $\hat{p}_j = k(z)$. When i and j share a vertex, the z -dependence of those diagrams are completely determined by the corresponding polarization vector and wavefunction. This class of diagrams contributes the factor $[k(z)|\epsilon^+(p(z))]$. We recall that $[k(z)| \propto z$ and $\epsilon^+(p(z)) \propto 1/z$, and so naively this term would not decay for large z . However, we point out that

$$[k(z)|\epsilon^+(p(z)) = \frac{[k(z) p]\langle q|}{\langle q p(z)\rangle} = \frac{[k p]\langle q|}{\langle q p(z)\rangle}, \tag{2.11}$$

so this type of diagram does vanish for large values of z . When the particles i and j are separated by any single (z -dependent) propagator, it is straightforward to check that similar cancellations occur in the numerator whenever an amplitude poses the threat of not decaying at large z . Finally, any further lengthening of the “ z -path” is harmless since the only components which grow with z are the scalar-gluon and the triple-gluon vertices. However, introducing such interactions must be accompanied by the appropriate bosonic propagator, which then compensates for the z behaviour of the vertex. Thus, we find that the choice $(+, +)$ is valid for an adjacent gluon-fermion pair (in that order), and similarly for the choice $(-, -)$.

The new element in our calculation is the continuation away from four dimensions. However, the continuation to D dimensions does not affect the analytic properties of $\mathcal{A}(z)$ or any of its ingredients, so the recursion relations remain valid, as do the choices for (i, j) given in the literature. This fact was already noted in [16]. We point out that the only qualitatively new ingredient in D dimensions is the helicity-flipping fermionic propagator. However, this in fact has even better large z behaviour than its standard helicity-preserving counterpart, since the numerator μ is constant, as opposed to \hat{H} which is linear in z .

⁸For D -dimensional momenta helicity is not well-defined, but one still has to sum over an appropriately chosen basis, as described above.

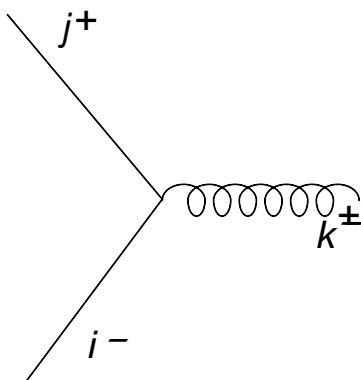


Figure 2: Gluon-Fermion primitive vertex, the fermion helicities are drawn, the gluon can have positive or negative helicity.

To summarize, in the following we will use the BCF recursion relations, always marking momenta in configurations that are proven to be allowed (that is, when the vanishing of the boundary term has been established, as described above).

2.5 Primitive vertices

We are interested in calculating tree level diagrams involving massless quarks and gluons, and massive scalars and fermions. Using the BCF recursion relations [10], we need as basic building blocks a few cubic amplitudes. We list here the vertices we need for our calculation. In every case these are simply constructed from the gauge theory action by contraction with the external wavefunction of the appropriate helicity. In all the cases listed below we treat all momenta as incoming. All vertices are written for (i, j, k) cyclically ordered, the expressions for the other cyclic ordering (j, i, k) differ by an overall sign if (i, j) are fermions.

The primitive vertices we need are:

- gluon-fermion vertex (figure 2):

$$\begin{aligned}
 \text{negative helicity gluon } k: & \quad \frac{\langle ik \rangle [qj]}{[qk]} \\
 \text{positive helicity gluon } k: & \quad -\frac{\langle iq \rangle [kj]}{\langle qk \rangle}
 \end{aligned}
 \tag{2.12}$$

where q is an arbitrary reference null vector with $q = |q\rangle[q|$. The vertex is independent of this choice as a consequence of gauge invariance. As each one of the primitive vertices is gauge invariant, these reference vectors can be chosen independently for each vertex.

- scalar-fermion vertex (figure 3):

$$\begin{aligned}
 \text{negative helicity external legs:} & \quad \langle ij \rangle \\
 \text{positive helicity external legs:} & \quad -[ij]
 \end{aligned}
 \tag{2.13}$$

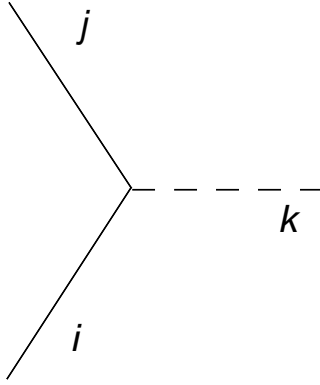


Figure 3: Scalar-Fermion primitive vertex, The 3 external legs have the same helicities.

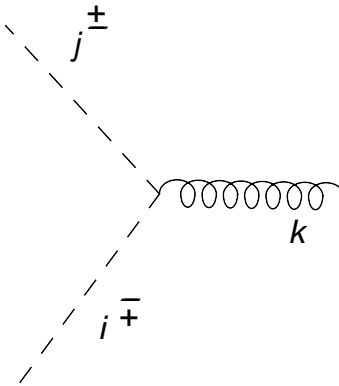


Figure 4: Gluon-scalar primitive vertex, the scalar helicities are drawn, the gluon can have positive or negative helicity.

- scalar-gluon vertex (figure 4):

$$\begin{aligned}
 \text{negative helicity gluon } k: & \quad -\frac{\langle k|j|q\rangle}{[qk]} \\
 \text{positive helicity gluon } k: & \quad \frac{\langle q|j|k\rangle}{\langle qk\rangle}
 \end{aligned}
 \tag{2.14}$$

- gluon cubic vertex (figure 5):

$$\begin{aligned}
 \text{MHV vertex:} & \quad \frac{\langle ij\rangle^3}{\langle jk\rangle\langle ki\rangle} \\
 \overline{\text{MHV}} \text{ vertex:} & \quad \frac{[ij]^3}{[jk][ki]}.
 \end{aligned}
 \tag{2.15}$$

There are other non-vanishing cubic vertices which we will not need, therefore we will not present them here. Additionally, for D-dimensional fermions one utilizes the full expression for the gluon polarization, resulting in additional terms in the interaction vertices which we write down when we use them below.

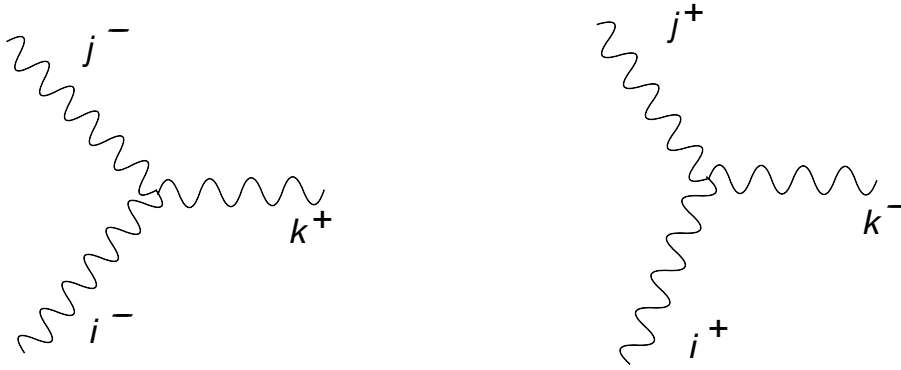


Figure 5: Gluon MHV (left) and \overline{MHV} (right) cubic vertices.

2.6 Checks on the amplitudes

Though the on-shell recursion relations are proven, it is still useful to perform a few checks on the resulting expression, verifying the various intermediate steps leading to the final expression. These steps include choosing marked momenta (such that $A(z)$ vanishes as $z \rightarrow \infty$), choices of various reference momenta, and straightforward (but sometimes tedious) algebra.

The first check one can perform is comparison with the result of Feynman diagram calculation. We have checked our expressions against such calculations for all the four point amplitudes and some of the five point ones. Typically the recursion relations yield much more compact expressions for the amplitudes, so the main complication is to reduce the complex Feynman diagram result to the simpler expression.

Another check we have performed is (some of) the collinear limits of the amplitudes. The collinear limits are a subset of the multi-particle poles which occur at tree level amplitudes. As the sum of two neighboring momenta becomes on-shell the amplitude factorizes in the appropriate channel. All our amplitudes have poles at the right location, and in some cases we have checked explicitly that the residue of the pole is the expected one. We exemplify the collinear limit in the appendix.

Finally, in the limit $\mu \rightarrow 0$ all external legs are four dimensional. In some cases the amplitudes are known in that limit, and we reproduce these results.

3. Four point amplitudes

The amplitudes with four external legs are fairly simple, and can be checked explicitly against Feynman diagram calculations. We present the details and the results in this section as a demonstration of the technique and the new issues arising when D-dimensional fermions and scalars are included.

In addition, all such amplitudes can be seen to have the correct factorization limits. Indeed, to see the singularity structure it is sufficient to inspect the denominators, we easily see that they vanish if and only if the sum of two adjacent momenta becomes on-shell (so

spurious singularities are absent). Checking factorization amount to verifying that the residue of these poles is the expected one.⁹

3.1 Type-A amplitudes

These amplitudes have two adjacent massive fermions (of momenta and helicities K_1^+, K_2^-) and two adjacent gluons (of momenta p_1, p_2). We discuss all helicity configurations in turn.

The first case of where the gluons are of opposite helicities is the amplitude $\mathcal{A}_4(K_1^+, K_2^-, p_1^+, p_2^-)$. We choose the marked momenta to be $(i, j) = (p_1^+, p_2^-)$, as this is one of the configurations for which there is a general proof of vanishing at infinity. In this case there is only one possible diagram appearing in the recursion relation, with intermediate fermion of momentum $P = K_2 + p_1$. One gets

$$\mathcal{A}_4(K_1^+, K_2^-, 1^+, 2^-) = \frac{\langle K_2 q_1 \rangle [\hat{1} - \hat{P}]}{\langle q_1 \hat{1} \rangle} \frac{1}{P^2} \frac{\langle \hat{P} \hat{2} \rangle [q_2 K_1]}{[q_2 \hat{2}]} \quad (3.1)$$

where q_1, q_2 are two reference momenta. Note that the helicity of the intermediate states of momentum P is determined by that of the external momenta.

We choose $q_1 = \hat{2}$ and $q_2 = \hat{1}$, and use $|\hat{1}\rangle = |1\rangle$ and $|\hat{2}\rangle = |2\rangle$, then this becomes

$$- \frac{\langle K_2 2 \rangle [1 | \not{p} | 2 \rangle [1 K_1]}{\langle 2 \hat{1} \rangle [1 \hat{2}] P^2} = - \frac{\langle K_2 2 \rangle [1 | k_2 | 2 \rangle [1 K_1]}{\langle 2 1 \rangle [1 2] P^2}. \quad (3.2)$$

This result is valid in D-dimensions, with states such as $|K_2\rangle$ are defined in section 2. Note also that in this case the relevant part of P appearing in the intermediate state is p , as $[1 | \not{\mu} | 2 \rangle = 0$ by chirality selection rules.

The next step we simplify the result by taking the gluons to be massless in four dimensions, and the fermion momenta to be massless in D-dimensions (with -2ϵ component μ). Then the four dimensional component of P is $p = p_1 + k_2$, and the -2ϵ component is μ . This result can be shown to be identical to the one obtained from using Feynman graphs

$$\mathcal{A}_4(K_1^+, K_2^-, 1^+, 2^-) = \frac{(\epsilon_1^+ \cdot k_2) [K_1 | \epsilon_2^+ | K_2]}{(p_1 + k_2)^2 - \mu^2} \quad (3.3)$$

A slight variation of the same calculation yields

$$\mathcal{A}_4(K_1^+, K_2^-, 1^-, 2^+) = - \frac{[K_1 2] \langle 1 K_2 \rangle [2 | K_1 | 1]}{P^2 \langle 1 2 \rangle [2 1]} = \frac{(\epsilon_1^- \cdot k_1) [K_1 | \epsilon_2^+ | K_2]}{(p_1 + k_2)^2 - \mu^2} \quad (3.4)$$

where the intermediate momentum $P = p_1 + K_2$ in the only diagram contributing to the recursion relations.

Additionally, for these gluon helicities there could be one helicity flipping amplitudes, using the same choices of marked and reference momenta one gets:

$$\mathcal{A}_4(K_1^+, K_2^+, 1^+, 2^-) = \frac{[K_2 1] \langle 2 - \hat{P} \rangle}{\langle 2 1 \rangle} \frac{1}{P^2} \frac{\langle \hat{P} 2 \rangle [1 K_1]}{[1 2]} \quad (3.5)$$

⁹The term collinear limit is inaccurate when massive momenta are involved, what we really mean is two-particle factorization limits.

However, using $|P\rangle\langle P| = \omega_- \not{\mu}$ and $\langle 2|\not{\mu}|2\rangle = 0$ this amplitude vanishes.

Now, if the gluons are of the same helicity, we get the amplitudes $\mathcal{A}_4(K_1, K_2, p_1^+, p_2^+)$, where the massive momenta K_1, K_2 are of either chirality. In this case both external states are of the form $|K_i\rangle = |K_i\rangle + |K_i\rangle$. Choosing the marked momenta to be $(i, j) = (p_1^+, p_2^+)$ gives one possible intermediate (D-dimensional fermionic) state with $P = K_2 + p_1$.

When contracting with the gluon polarizations $\not{\epsilon}^+$ one gets the interaction vertex

$$\frac{1}{\langle q\hat{1}\rangle} \left(\langle \hat{P}q \rangle [\hat{1}K_2] + [\hat{P}\hat{1}] \langle qK_2 \rangle \right) \quad (3.6)$$

with a similar expression for the other interaction vertex in the diagram. We get

$$\begin{aligned} \mathcal{A}_4(K_1, K_2, 1^+, 2^+) &= \frac{1}{\langle q_1\hat{1}\rangle} \left(\langle \hat{P}q_1 \rangle [\hat{1}K_2] + [\hat{P}\hat{1}] \langle q_1K_2 \rangle \right) \frac{1}{P^2} \\ &\quad \frac{1}{\langle q_2\hat{2}\rangle} \left(\langle -\hat{P}q_2 \rangle [\hat{2}K_1] + [-\hat{P}\hat{2}] \langle q_2K_1 \rangle \right). \end{aligned} \quad (3.7)$$

There are four terms, corresponding to the four possible helicity assignments for K_1, K_2 . As the amplitude vanishes in four dimensions, we know that only helicity flipping terms has to be retained. Starting with K_1^-, K_2^- we get (choosing $q_1 = \hat{2}$ and $q_2 = \hat{1}$),

$$\mathcal{A}_4(K_1^-, K_2^-, 1^+, 2^+) = - \frac{[\hat{P}\hat{1}] \langle \hat{2}K_2 \rangle [\hat{P}\hat{2}] \langle \hat{1}K_1 \rangle}{\langle 21 \rangle \langle 12 \rangle} \quad (3.8)$$

using $|P\rangle\langle P| = \omega_+ \not{\mu}$, and the more standard projections gives

$$- \frac{\langle K_2 | \not{2} \not{\mu} \not{1} | K_1 \rangle}{\langle 21 \rangle \langle 12 \rangle}. \quad (3.9)$$

As $\not{\mu}$ anti-commutes with $\not{1}, \not{2}$, those combine to give $(1+2)^2 - \not{1}\not{2}$. The last term gives vanishing contribution (using e.g. $[P|\not{2}|K_1\rangle = 0$ by momentum conservation), therefore

$$\mathcal{A}_4(K_1^-, K_2^-, 1^+, 2^+) = \frac{[12]}{\langle 12 \rangle} \frac{\langle K_2 | \not{\mu} | K_1 \rangle}{(k_2 + p_1)^2 - \mu^2}. \quad (3.10)$$

This matches the result quoted in [29]. Similarly for the last helicity configuration one obtains

$$\mathcal{A}_4(K_1^+, K_2^+, 1^+, 2^+) = \frac{[12]}{\langle 12 \rangle} \frac{[K_2 | \not{\mu} | K_1]}{(k_2 + p_1)^2 - \mu^2}. \quad (3.11)$$

3.2 Type-B amplitudes

These amplitudes have two massive scalars (of opposite helicities) with momenta L_1, L_2 , and two massless fermions of opposite helicities, and momenta $k_1 = \lambda_1 \tilde{\lambda}_1$ and $k_2 = \lambda_2 \tilde{\lambda}_2$. As we are interested always in adjacent massive legs, the only non-vanishing helicity preserving configurations have cyclic ordering of momenta $(k_1^+, k_2^-, L_1^-, L_2^+)$ or $(k_1^+, k_2^-, L_1^+, L_2^-)$. The helicity violating configuration is $(k_1^+, k_2^+, L_1^+, L_2^+)$.

For all these configurations we choose the two marked momenta to be $(i, j) = (k_1, L_1)$. In this case there are two possible grouping of momenta, or two possible diagrams in the

recursion relation. In one of them the intermediate momentum is $P = k_1 + L_2$, and the intermediate state is a D-dimensional fermion. In the other diagram the intermediate momentum is $q = k_1 + k_2$, and the intermediate state is a gluon which can be of positive or negative helicity. The first set of diagrams can lead to helicity flipping via the D-dimensional internal fermion, whereas the diagrams with an internal gluon only lead to helicity conserving amplitudes.

The first amplitude $\mathcal{A}_4(k_1^+, k_2^-, L_1^-, L_2^+)$ can be written as a sum of three terms

$$\mathcal{A}_4(k_1^+, k_2^-, L_1^-, L_2^+) = -\frac{[\hat{k}_1 - \hat{P}][\hat{P}k_2]}{P^2} - \frac{\langle k_2 q_1 \rangle [\hat{k}_1 - \hat{q}]}{\langle q_1 - \hat{q} \rangle} \frac{1}{Q^2} \frac{\langle \hat{q} | l_2 | q_2 \rangle}{[q_2 \hat{q}]} + \frac{\langle k_2 \hat{q} \rangle [q_3 \hat{k}_1]}{[q_3 \hat{q}]} \frac{1}{Q^2} \frac{\langle q_4 | l_2 | \hat{q} \rangle}{\langle q_4 \hat{q} \rangle}.$$

The last term vanishes if we choose $q_3 = k_1$ (note that $\langle k_1 \hat{q} \rangle \neq 0$). For the middle term we choose $q_1 = q_2 = k_1$, the amplitude is then

$$\frac{1}{q^2} \left\{ [k_1 | l_2 | k_2] + \frac{[\hat{k}_1 | \hat{q} l_2 k_1 | k_2]}{\langle k_1 \hat{q} \rangle [k_1 \hat{q}]} \right\} \quad (3.12)$$

using elementary consideration the numerator can be simplified

$$[\hat{k}_1 | \hat{q} l_2 k_1 | k_2] = -2(k_1 \cdot k_2) [k_1 | l_2 | k_2] = -2(k_1 \cdot \hat{q}) [k_1 | l_2 | k_2] \quad (3.13)$$

therefore

$$\mathcal{A}_4(k_1^+, k_2^-, L_1^-, L_2^+) = [k_1 | l_2 | k_2] \left\{ \frac{1}{(k_1 + l_2)^2 - \mu^2} + \frac{1}{(k_1 + k_2)^2} \right\} \quad (3.14)$$

which can be easily checked to be the result obtained from Feynman diagrams.

Using similar reasoning, the the other helicity conserving amplitude evaluates to be

$$\mathcal{A}_4(k_1^+, k_2^-, L_1^+, L_2^-) = \frac{[k_1 | l_2 | k_2]}{(k_1 + k_2)^2}. \quad (3.15)$$

Note the absence of singularity as $k_1 + L_2$ becomes on-shell, as there is no appropriate helicity assignment for the would-be on-shell intermediate state. This result can be easily checked to be the result of a sum of two Feynman diagrams.

As before, the intermediate state is effectively four dimensional for helicity preserving external states. We now evaluate the helicity flipping amplitude, for them the only possible intermediate state is fermionic and its propagator is effectively $\not{\mu}$, which simplifies the calculation. The primitive vertex coupling scalars to fermions is unmodified, giving:

$$\mathcal{A}_4(k_1^+, k_2^+, L_1^+, L_2^+) = \frac{[\hat{k}_1 - \hat{P}][\hat{P}k_2]}{P^2} = \frac{-[k_1 | \not{\mu} | k_2]}{(k_1 + l_2)^2 - \mu^2} \quad (3.16)$$

However, for the purpose those type B amplitudes where the ‘‘helicity’’ of the scalars is flipped are not used as ingredient in the one-loop calculation we are ultimately interested in. We therefore only consider the case where the two massive scalars have opposite ‘‘helicity’’. This is useful because often this eliminates the fermion helicity-flipping terms as well.

3.3 Type-C amplitudes

There are a few helicity configurations relevant here. We will sketch the calculation and give the results. In all cases we take the fermionic momenta to be k_1, K_2 , the scalar momenta L and the gluon momentum p .

For $\mathcal{A}_4(k_1^+, K_2^+, L^+, p^+)$ we choose the marked momenta to be $(i, j) = (p^+, k_1^+)$. There is one possible grouping of momenta for which the intermediate state is a D-dimensional scalar of momentum $Q = p + L$. The recursion relation reads

$$\mathcal{A}_4(k_1^+, K_2^+, L^+, p^+) = -\frac{\langle q|L|\hat{p}\rangle}{\langle q\hat{p}\rangle} \frac{1}{Q^2} [\hat{k}_1 K_2] \quad (3.17)$$

choosing $q = \hat{k}_1$ gives

$$\frac{\langle k_1|\not{l}|p\rangle[K_2\hat{k}_1]}{\langle k_1p\rangle Q^2} \quad (3.18)$$

where we already projected onto the appropriate component of \not{l} , namely l .

The last step involved calculating $[K_2\hat{k}_1]$ which will be used repeatedly below. Using the result

$$[K_2\hat{k}_1] = \frac{[p|\not{l}k_2 + \mu^2|K_2]}{\langle k_1|\not{l}|p\rangle} \quad (3.19)$$

one finally gets

$$\mathcal{A}_4(k_1^+, K_2^+, L^+, p^+) = \frac{[p|\not{l}k_2 + \mu^2|K_2]}{\langle k_1p\rangle [(p+l)^2 - \mu^2]}. \quad (3.20)$$

For the amplitude $\mathcal{A}_4(k_1^+, K_2^+, L^+, p^-)$ we choose the marked momenta $(i, j) = (k_1^+, p^-)$, resulting in similar calculation (where the helicity of the external gluon is flipped). The result is

$$\mathcal{A}_4(k_1^+, K_2^+, L^+, p^-) = \frac{[k_1 K_2] \langle p|\not{l}|k_1\rangle}{[k_1p] [(p+l)^2 - \mu^2]}. \quad (3.21)$$

Note that for these two amplitudes, where the fermions are adjacent, the intermediate state is massive scalar and consequently there is no helicity-flipping intermediate state.

For the amplitude $\mathcal{A}_4(k_1^+, p^+, K_2^+, L^+)$ we choose $(i, j) = (p^+, k_1^+)$, therefore the intermediate state is of momentum $Q = p + K_2$, and

$$\mathcal{A}_4(k_1^+, p^+, K_2^+, L^+) = -\frac{[K_2\hat{p}]\langle q - \hat{Q}\rangle}{\langle q\hat{p}\rangle} \frac{1}{Q^2} [\hat{Q}\hat{k}_1] \quad (3.22)$$

choosing $q = k_1$ gives after some elementary algebra

$$\mathcal{A}_4(k_1^+, p^+, K_2^+, L^+) = \frac{\mu^2}{\langle k_1p\rangle \langle pK_2\rangle}. \quad (3.23)$$

As the intermediate state in this amplitude is D-dimensional fermion, there is a similar amplitude with opposite helicity for one of the fermions, namely

$$\mathcal{A}_4(k_1^+, p^+, K_2^-, L^+) = \frac{\langle K_2q\rangle[\hat{p}\hat{Q}]}{\langle \hat{p}q\rangle} \frac{1}{P^2} [\hat{P}\hat{k}_1] = \frac{\langle K_2k_1\rangle[p|\not{l}|\hat{k}_1]}{\langle p k_1\rangle Q^2} \quad (3.24)$$

after calculating $[p|\not{\mu}|\hat{k}_1]$ we end up with the amplitude

$$\mathcal{A}_4(k_1^+, p^+, K_2^-, L^+) = \frac{\langle K_2 k_1 \rangle [p|\not{\mu}|k_1]}{\langle p k_1 \rangle [(p+k_2)^2 - \mu^2]}. \quad (3.25)$$

Finally for the amplitude $\mathcal{A}_4(k_1^+, p^-, K_2^+, L^+)$ we choose $(i, j) = (k_1^+, p^-)$, giving

$$\mathcal{A}_4(k_1^+, p^-, K_2^+, L^+) = -[\hat{k}_1 - \hat{P}] \frac{1}{P^2} \frac{\langle \hat{P} \hat{p} \rangle [q K_2]}{[q \hat{p}]} \quad (3.26)$$

with $P = K_2 + p$, choosing $q = k_1$ gives

$$\mathcal{A}_4(k_1^+, p^-, K_2^+, L^+) = -\frac{[k_1 K_2]}{[k_1 p]} \frac{\langle p|\not{\mu}|k_1 \rangle}{(k_1 + l)^2 - \mu^2}. \quad (3.27)$$

Once again, since the intermediate state has helicity flipping part, we could get the additional amplitude

$$\mathcal{A}_4(k_1^+, p^-, K_2^-, L^+) = [\hat{k}_1 - \hat{P}] \frac{1}{P^2} \frac{\langle K_2 \hat{p} \rangle [q \hat{P}]}{[q \hat{p}]} \quad (3.28)$$

However, choosing again $q = k_1$, and using $[k_1|\not{\mu}|k_1] = 0$, this amplitude vanishes.

4. Five point amplitudes

We list below the results of the calculation of the relevant five point amplitudes, and the checks they satisfy. In all cases the calculation follows the lines of the corresponding four point amplitude calculation, and we omit the details for brevity.

4.1 Type-A amplitudes

These amplitudes include two adjacent D-dimensional fermions of momenta K_1, K_2 , and three adjacent gluons of momenta p_1, p_2, p_3 . The results are

$$\mathcal{A}_5(K_1^-, K_2^-, p_1^+, p_2^+, p_3^+) = \frac{\langle K_2|\not{\mu}|K_1 \rangle \langle K_2|(k_1 + k_2)|3 \rangle}{\langle K_2 1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3 K_1 \rangle [K_1 3]}.$$

There is no singularity as $K_1 + K_2$ becomes null, since no helicity assignment exists for an intermediate state. The result is symmetric when exchanging (1, 3) and (K_1, K_2) , therefore there are only two collinear limits to check. We perform these checks in the appendix as a demonstration.

Note also that the amplitude vanishes in four dimensions (setting $\mu = 0$) as it should, and therefore only helicity-flipping parts exist. The other possible helicity assignment for fermions gives

$$\mathcal{A}_5(K_1^+, K_2^+, p_1^+, p_2^+, p_3^+) = \frac{[K_1|(p_3 + k_1)\not{\mu}|2 \rangle \langle K_2|(k_1 + k_2)|3 \rangle}{\langle K_2 1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3 K_1 \rangle [K_1 3]}.$$

The rest of the type A amplitudes are given as sum over two diagrams, one with internal gluon and one with internal D-dimensional fermion

$$\begin{aligned}
 \mathcal{A}_5(K_1^+, K_2^-, p_1^+, p_2^-, p_3^+) &= -\frac{\langle K_2 2 \rangle^3 \langle K_1 2 \rangle [3 K_1]}{\langle K_2 1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3 K_1 \rangle [3] (k_{\not{1}} + k_{\not{2}}) | K_2 \rangle} \\
 \mathcal{A}_5(K_1^-, K_2^-, p_1^+, p_2^-, p_3^+) &= -\frac{[13]^4 \langle K_1 | \not{\mu} | K_2 \rangle}{[12][23][K_1 K_2] \langle K_2 K_1 \rangle [3] (k_{\not{1}} + k_{\not{2}}) | K_2 \rangle} - \\
 &\quad -\frac{\langle K_2 2 \rangle \langle 2 K_1 \rangle [3] k_{\not{1}} | 2 \rangle [1 | \not{\mu} | 3]}{\langle 21 \rangle (K_2 + p_1)^2 (p_2 + p_3)^2 (K_1 + p_3)^2} \\
 \mathcal{A}_5(K_1^+, K_2^+, p_1^+, p_2^-, p_3^+) &= -\frac{[13]^4 [K_1 | \not{\mu} | K_2]}{[12][23][K_1 K_2] \langle K_2 K_1 \rangle [3] (k_{\not{1}} + k_{\not{2}}) | K_2 \rangle} \\
 \mathcal{A}_5(K_1^+, K_2^-, p_1^+, p_2^+, p_3^-) &= \frac{\langle K_2 3 \rangle^2 [K_1 | (k_{\not{1}} + k_{\not{2}}) | 3]}{\langle 12 \rangle \langle 23 \rangle \langle K_1 K_2 \rangle [K_1 | (k_{\not{1}} + k_{\not{2}}) | 1]} + \\
 &\quad + \frac{\mu^2 [12] \langle K_2 3 \rangle [2 K_1]}{\langle 12 \rangle [23] (K_2 + p_1)^2 (K_1 + p_3)^2} \\
 \mathcal{A}_5(K_1^+, K_2^+, p_1^+, p_2^+, p_3^-) &= \frac{[12][2 K_1]^2 [K_1 | \not{\mu} | K_2]}{\langle K_2 1 \rangle \langle K_2 1 \rangle [23][K_1 3] [K_1 | (k_{\not{2}} + k_{\not{1}}) | 1]} \\
 \mathcal{A}_5(K_1^+, K_2^-, p_1^+, p_2^-, p_3^-) &= \frac{\langle 1 K_2 \rangle^2 [K_1 | (k_{\not{1}} + k_{\not{2}}) | 1] [K_2 | (k_{\not{1}} + k_{\not{2}}) | 1]}{\langle K_2 1 \rangle \langle 12 \rangle \langle 23 \rangle \langle K_1 K_2 \rangle [K_2 K_1] [K_2 | (k_{\not{1}} + k_{\not{2}}) | 3]} \\
 \mathcal{A}_5(K_1^+, K_2^+, p_1^+, p_2^-, p_3^-) &= \frac{[23] [K_2 2]^2 \langle K_2 1 \rangle [K_1 | \not{\mu} | K_2]}{(p_3 + K_1)^2 (K_2 + p_1)^2 [K_2 | (k_{\not{1}} + k_{\not{2}}) | 3]}
 \end{aligned}$$

4.2 Type-B amplitudes

These amplitudes have two massive scalars (of opposite helicities) with momenta L_1, L_2 , and two massless fermions of momenta k_1, k_2 , and an additional gluon of momentum p . The helicity flipping part of the fermion propagator are less relevant in this set of calculations, since fermion helicities are correlated with that of the external scalars. The results are

$$\begin{aligned}
 \mathcal{A}_5(p^+, L_1^-, L_2^+, k_1^+, k_2^-) &= \frac{\langle k_2 | l_{\not{1}} | p \rangle [k_1 | l_{\not{2}} | k_2 \rangle}{\langle k_2 p \rangle \langle p | l_{\not{1}} | p \rangle} \left(\frac{\langle k_2 | l_{\not{1}} | p \rangle}{[k_1 | (l_{\not{2}} l_{\not{1}} + \mu^2) | p] \langle k_1 k_2 \rangle} + \frac{1}{(k_1 + L_2)^2} \right) \\
 &\quad + \frac{\mu^2 [p k_1]^3}{[k_1 | (l_{\not{2}} l_{\not{1}} + \mu^2) | p] [k_2 k_1] (L_1 + L_2)^2} \\
 \mathcal{A}_5(p^-, L_1^-, L_2^+, k_1^+, k_2^-) &= \left(\frac{\langle p | l_{\not{2}} | k_2 \rangle}{\langle p | (l_{\not{1}} l_{\not{2}} + \mu^2) | k_1 \rangle [k_1 k_2]} + \frac{1}{(k_2 + L_2)^2} \right) \times \\
 &\quad \times \left(\frac{[k_1 | l_{\not{1}} | p] [k_1 | l_{\not{2}} | k_2 \rangle - \mu^2 [12] \langle 2 p \rangle}{\langle p | l_{\not{1}} | p \rangle [p 2]} \right) \\
 &\quad + \frac{\mu^2 \langle p k_2 \rangle^2 \langle p k_1 \rangle}{\langle p | (l_{\not{1}} l_{\not{2}} + \mu^2) | k_2 \rangle \langle k_1 k_2 \rangle (L_1 + L_2)^2} \\
 \mathcal{A}_5(p^+, L_1^-, L_2^+, k_1^-, k_2^+) &= \frac{\langle k_2 | l_{\not{1}} | p \rangle^2 \langle k_1 | l_{\not{2}} | k_2 \rangle}{\langle k_1 k_2 \rangle \langle k_2 p \rangle \langle p | l_{\not{1}} | p \rangle [k_1 | (l_{\not{2}} l_{\not{1}} + \mu^2) | p]} \\
 &\quad + \frac{\mu^2 [p k_2]^2 [p k_1]}{[k_1 k_2] [k_1 | (l_{\not{2}} l_{\not{1}} + \mu^2) | p] (L_1 + L_2)^2} \\
 \mathcal{A}_5(p^-, L_1^-, L_2^+, k_1^-, k_2^+) &= -\frac{\langle p | l_{\not{1}} | k_2 \rangle^2 \langle k_1 | l_{\not{2}} | k_2 \rangle}{[k_1 k_2] [k_2 p] \langle p | l_{\not{1}} | p \rangle \langle p | (l_{\not{1}} l_{\not{2}} + \mu^2) | k_1 \rangle}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_5(L_1^+, p^+, L_2^-, k_1^-, k_2^+) &= \frac{\mu^2 \langle pk_1 \rangle^3}{\langle k_1 k_2 \rangle \langle p | (\not{l} \not{l}_2 + \mu^2) | k_1 \rangle (L_1 + L_2)^2} \\
 &+ \frac{\langle k_2 | \not{l} \not{l}_2 | k_1 \rangle}{\langle k_1 p \rangle \langle p k_2 \rangle} \left(\frac{\langle k_1 k_2 \rangle}{\langle k_2 | \not{l} \not{l}_2 | k_1 \rangle} + \frac{1}{(L_1 + L_2)^2} \right) \\
 &+ \frac{\mu^2 [p | \not{l} | k_2]^2}{\langle k_2 | \not{l} \not{l}_2 | k_1 \rangle (L_1 + L_2)^2 (L_1 + k_2)^2} \\
 \mathcal{A}_5(L_1^-, p^+, L_2^+, k_1^-, k_2^+) &= \frac{\langle k_2 | \not{l} \not{l}_2 | k_1 \rangle}{\langle k_1 p \rangle \langle p k_2 \rangle (L_1 + L_2)^2}
 \end{aligned}$$

We also find that all amplitude where fermion helicity is flipped are vanishing.

4.3 Type-C amplitudes

These consist of massless fermion of momentum k_1 , massive fermion of momentum K_2 , massive scalar of momentum L and two massless gluons of momenta p_1, p_2 . The results are:

$$\begin{aligned}
 \mathcal{A}_5(p_1^+, p_2^+, k_1^+, L^+, K_2^+) &= -\frac{\mu^2}{\langle K_2 1 \rangle \langle 1 2 \rangle \langle 2 k_1 \rangle} \\
 \mathcal{A}_5(p_1^+, p_2^+, k_1^+, L^+, K_2^-) &= \frac{\langle K_2 | k_1 \not{\mu} (\not{l} + \not{k}_2) | K_2 \rangle}{\langle K_2 1 \rangle \langle 1 2 \rangle \langle 2 k_1 \rangle (k_1 + L)^2} \\
 \mathcal{A}_5(p_1^+, p_2^-, k_1^+, L^+, K_2^+) &= -\frac{\langle 2 K_2 \rangle [k_1 | \not{\mu} | 2]^2}{\langle K_2 1 \rangle \langle 1 2 \rangle [k_1 | (\not{l} + \not{k}_2) | K_2] (k_1 + L)^2} \\
 &\quad - \frac{\mu^2 [k_1 1]^3}{[1 2][2 k_1] [k_1 | (\not{l} + \not{k}_2) | K_2] (k_1 + L)^2} \\
 \mathcal{A}_5(p_1^+, p_2^-, k_1^+, L^+, K_2^-) &= \frac{\langle K_2 2 \rangle^2 [1 | \not{\mu} | k_1] \langle 2 | \not{\mu} | k_1 \rangle}{\langle K_2 1 \rangle \langle 1 2 \rangle [k_1 | \not{\mu} | K_2] [K_2 1] (k_1 + L)^2} \\
 &\quad + \frac{\langle K_2 | (\not{l} + \not{k}_2) | 1 \rangle [1 | \not{\mu} | k_1] [1 k_1]^2}{[K_2 1][1 2][2 k_1] [k_1 | (\not{l} + \not{k}_2) | K_2] (K_2 + L)^2} \\
 \mathcal{A}_5(p_1^-, p_2^+, k_1^+, L^+, K_2^+) &= \frac{\langle 1 k_1 \rangle \langle 1 | \not{\mu} + \not{k}_2 | K_2 \rangle^2}{\langle 1 2 \rangle \langle 2 k_1 \rangle \langle k_1 | (\not{l} + \not{k}_2) | K_2 \rangle (K_2 + L)^2} \\
 &\quad - \frac{\mu^2 [K_2 2]^3}{[1 2][K_2 1] [K_2 | (\not{l} + \not{k}_2) | k_1] (k_1 + L)^2} \\
 \mathcal{A}_5(p_1^-, p_2^+, k_1^+, L^+, K_2^-) &= \frac{\langle K_1 | \not{\mu} | 2 \rangle [2 | \not{\mu} | k_1] [K_2 2]}{[K_2 | (\not{l} + \not{k}_2) | k_1] [K_2 1] [1 2] (k_1 + L)^2} \\
 \mathcal{A}_5(p_1^-, p_2^-, k_1^+, L^+, K_2^+) &= \frac{[K_2 k_1] [k_1 | \not{\mu} \not{\mu} + (k_1 + L)^2 | K_2]}{[1 2][K_2 1][2 k_1] (k_1 + L)^2} \\
 \mathcal{A}_5(p_1^+, k_1^+, p_2^+, L^+, K_2^+) &= -\frac{\mu^2 \langle k_1 | \not{\mu} | 2 \rangle}{\langle K_2 1 \rangle \langle 1 k_1 \rangle \langle k_1 2 \rangle \langle 2 | \not{\mu} | 2 \rangle} \\
 \mathcal{A}_5(p_1^+, k_1^+, p_2^+, L^+, K_2^-) &= -\frac{\langle K_2 k_1 \rangle [1 | \not{\mu} (k_2 \not{l} + \mu^2) | 2]}{\langle K_2 1 \rangle \langle 1 k_1 \rangle \langle k_1 2 \rangle \langle 2 | \not{\mu} | 2 \rangle [1 K_2]} \\
 \mathcal{A}_5(p_1^+, k_1^+, p_2^-, L^+, K_2^+) &= \frac{\langle 2 K_2 \rangle [K_2 | \not{\mu} | 2] [K_2 | \not{\mu} + \not{k}_2 | 2]}{\langle 1 k_1 \rangle \langle 2 | (\not{l} \not{k}_2 + \mu^2) | 1 \rangle (K_2 + L)^2} \\
 &\quad + \frac{\mu^2 \langle 2 | \not{\mu} | k_1 \rangle^2}{[k_1 2] \langle K_2 1 \rangle \langle 2 | \not{\mu} | 2 \rangle \langle 2 | (\not{l} \not{k}_2 + \mu^2) | 1 \rangle}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_5(p_1^+, k_1^+, p_2^-, L^+, K_2^-) &= \frac{\langle K_2 | ((k_2 \not{\epsilon} + \not{l}) \not{l} + \mu^2) | 2 \rangle [1 | \not{\mu} | k_1] \langle 2 | \not{l} | k_1 \rangle}{\langle 1 | (k_2 \not{\epsilon} + \mu^2) | 2 \rangle \langle 1 K_2 \rangle [k_2 1] [k_1 2] \langle 2 | \not{l} | 2 \rangle} \\
 \mathcal{A}_5(p_1^-, k_1^+, p_2^+, L^+, K_2^+) &= \frac{[2 | \not{l} (k_1 \not{\epsilon} + \not{z}) | K_2]^2}{[K_2 1] \langle k_1 2 \rangle \langle 2 | \not{l} | 2 \rangle [1 | (k_2 \not{\epsilon} + \mu^2) | 2]} \\
 &\quad + \frac{[k_1 2]^2 [2 | (l k_2 \not{\epsilon} + \mu^2) | K_2]}{[1 k_1] [2 | (l k_2 \not{\epsilon} + \mu^2) | 1] (K_2 + L)^2} \\
 \mathcal{A}_5(p_1^-, k_1^+, p_2^-, L^+, K_2^+) &= \frac{[K_2 k_1]^2 \langle 2 | \not{l} | k_1 \rangle}{[K_2 1] [1 k_1] [k_1 2] \langle 2 | \not{l} | 2 \rangle} \\
 \mathcal{A}_5(k_1^+, p_1^+, p_2^+, L^+, K_2^+) &= \frac{[2 | (l k_2 \not{\epsilon} + \mu^2) | k_1] \langle k_1 | k_2 | K_2 \rangle}{\langle k_1 1 \rangle \langle 1 2 \rangle \langle 2 | \not{l} | 2 \rangle [(K_2 + k_1)^2 - \mu^2]} \\
 \mathcal{A}_5(k_1^+, p_1^+, p_2^+, L^+, K_2^-) &= \frac{\langle 1 | \not{l} | 2 \rangle \langle K_2 k_1 \rangle [2 | (l k_2 \not{\epsilon} + \mu^2) | k_1]}{\langle k_1 1 \rangle \langle 1 2 \rangle \langle 2 | \not{l} | 2 \rangle [2 | (l k_2 \not{\epsilon} + \mu^2) | K_2] \langle K_2 1 \rangle} \\
 \mathcal{A}_5(k_1^+, p_1^+, p_2^-, L^+, K_2^+) &= \frac{\langle k_1 2 \rangle [K_2 | (l + k_2 \not{\epsilon}) | 2] \langle 2 | \not{l} | K_2 \rangle \langle K_2 2 \rangle}{\langle k_1 1 \rangle \langle 1 2 \rangle \langle 2 | (l k_2 \not{\epsilon} + \mu^2) | k_1 \rangle (K_2 + L)^2} \\
 &\quad + \frac{\langle k_1 | k_2 | K_2 \rangle [k_1 1] \langle 2 | \not{l} | 1 \rangle^2}{[1 2] [2 | \not{l} | 2] \langle k_1 | (k_2 \not{\epsilon} + \mu^2) | 2 \rangle (k_1 + K_2)^2} \\
 \mathcal{A}_5(k_1^+, p_1^+, p_2^-, L^+, K_2^-) &= \frac{\langle K_2 k_1 \rangle [1 | \not{\mu} | k_1] \langle 2 | \not{l} | 1 \rangle^3}{[K_2 1] [1 2] [2 | \not{l} | 2] \langle 2 | \not{l} (\not{l} + \not{z}) | K_2 \rangle \langle 2 | (l k_2 \not{\epsilon} + \mu^2) | k_1 \rangle} \\
 \mathcal{A}_5(k_1^+, p_1^-, p_2^+, L^+, K_2^+) &= \frac{\langle 1 | \not{l} | 2 \rangle^2 [k_1 K_2]^2 \langle 1 K_2 \rangle}{\langle 1 2 \rangle \langle 2 | \not{l} | 2 \rangle [k_1 | (k_2 \not{\epsilon} + \mu^2) | 2] [(k_1 + K_2)^2 - \mu^2]} \\
 &\quad + \frac{[2 | (l k_2 \not{\epsilon} + \mu^2) | K_2] [2 k_1]^3}{[k_1 1] [1 2] [2 | (l k_2 \not{\epsilon} + \mu^2) | k_1] (K_2 + L)^2} \\
 \mathcal{A}_5(k_1^+, p_1^-, p_2^+, L^+, K_2^-) &= \frac{\langle K_2 | (k_2 \not{\epsilon} + \not{l}) | 2 \rangle [2 | \not{\mu} | k_1] [2 k_1]^2}{[k_1 1] [1 2] [2 | l k_2 \not{\epsilon} + \mu^2 | k_1] (K_2 + L)^2} \\
 \mathcal{A}_5(k_1^+, p_1^+, p_2^+, L^+, K_2^+) &= \frac{[k_1 K_2]^2 \langle 2 | \not{l} (\not{l} + \not{z}) | K_2 \rangle}{[k_1 1] [1 2] [2 | \not{l} | 2] [(k_1 + K_2)^2 - \mu^2]}
 \end{aligned}$$

To summarize, we have applied the BCF recursion relations to amplitudes which involve D-dimensional fermions and scalars. These D-dimensional particles behave in most respects as their massive counterparts in four dimensions. We have used this formalism to obtain four and five point amplitudes where two of the particles have been continued away from four dimensions and the remaining particles are (on-shell) gluons and massless fermions. Our results possess expected factorization properties and have passed other consistency checks, as outlined in section 2.6. The tree-level amplitudes we have computed here are the building blocks (to be assembled using generalized unitarity in D dimensions) of the rational (nonsupersymmetric) terms of the one-loop amplitude of two (massless) fermions and up to three gluons. We leave this task for future work [27].

Acknowledgments

We are supported by National Science and Engineering Research Council of Canada.

A. Collinear limits

Let us discuss the factorization limits of the amplitude

$$\mathcal{A}_5(K_1^-, K_2^-, p_1^+, p_2^+, p_3^+) = \frac{\langle K_2 | \not{k} | K_1 \rangle \langle K_2 | (k_1' + k_2') | 3 \rangle}{\langle K_2 1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3 K_1 \rangle [K_1 3]}$$

As mentioned in the text, there is no singularity in the channel where $K_1 + K_2$ becomes on shell. In addition there is a symmetry of exchanging (1, 3) and (K_1, K_2) . This leaves two channels to check, when $p_1 + p_2$ becomes null, or when $p_3 + K_1$ becomes null in D-dimensions (so it approaches the mass shell condition $(p_3 + k_1)^2 = \mu^2$ in four dimensions).

For the first limit we denote $p = p_1 + p_2$, the amplitude as p becomes light-like factorizes to

$$\mathcal{A}_5(K_1^-, K_2^-, p_1^+, p_2^+, p_3^+) \rightarrow \mathcal{A}_3(p_1^+, p_2^+, -p^-) \frac{1}{p^2} \mathcal{A}_4(p^+, p_3^+, K_1^-, K_2^-)$$

The four point amplitude is of type A, so we get:

$$\mathcal{A}_5(K_1^+, K_2^-, p_1^+, p_2^+, p_3^+) \rightarrow \frac{[12]^3}{[1p][p2]} \frac{1}{\langle 12 \rangle [21]} \frac{[p3]}{\langle p3 \rangle} \frac{\langle K_1 | \not{k} | K_2 \rangle}{(K_1 + p_3)^2}$$

To get to the right form we multiply both numerator and denominator by $\langle p K_2 \rangle$, and use momentum conservation to eliminate p , for example

$$[1p] \langle p3 \rangle = [1p|3] = [1|2|3] = [12] \langle 23 \rangle$$

this results in

$$\mathcal{A}_5(K_1^+, K_2^-, p_1^+, p_2^+, p_3^+) \rightarrow \frac{[12]^3}{[12][12]} \frac{1}{\langle 12 \rangle [21]} \frac{\langle K_2 | (k_1' + k_2') | 3 \rangle \langle K_1 | \not{k} | K_2 \rangle}{\langle K_2 1 \rangle \langle 23 \rangle (K_1 + p_3)^2}$$

which coincides indeed with the factorization limit of the exact expression calculated (generally there are additional terms in the exact expression which are non-singular in the limit).

The second factorization limit is for the channel $Q = p_3 + K_1$, in that limit

$$\mathcal{A}_5(K_1, K_2, p_1^+, p_2^+, p_3^+) \rightarrow \mathcal{A}_3(p_3^+, K_1, -Q) \frac{1}{Q^2} \mathcal{A}_4(Q, K_2, p_1^+, p_2^+)$$

where Q is the momentum of a D-dimensional intermediate fermion. However physical amplitudes such as $\mathcal{A}_5(K_1, K_2, p_1^+, p_2^+, p_3^+)$ are to be assembled from their components such as $\mathcal{A}_5(K_1^\pm, K_2^\pm, p_1^+, p_2^+, p_3^+)$ which do not separately obey factorization constraints. We therefore did not check these more complicated factorization limits involving massive intermediate momentum.

References

- [1] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [[hep-th/0312171](#)].

- [2] V.P. Nair, *A current algebra for some gauge theory amplitudes*, *Phys. Lett.* **B 214** (1988) 215.
- [3] S.J. Parke and T.R. Taylor, *An amplitude for n gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.
- [4] R. Penrose and M.A.H. MacCallum, *Twistor theory: an approach to the quantization of fields and space-time*, *Phys. Rept.* **6** (1972) 241.
- [5] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [[hep-th/0403047](#)].
- [6] C.J. Zhu, *The googly amplitudes in gauge theory*, *JHEP* **04** (2004) 032 [[hep-th/0403115](#)];
 G. Georgiou and V.V. Khoze, *Tree amplitudes in gauge theory as scalar MHV diagrams*, *JHEP* **05** (2004) 070 [[hep-th/0404072](#)];
 G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *JHEP* **07** (2004) 048 [[hep-th/0407027](#)];
 J.B. Wu and C.J. Zhu, *Mhv vertices and scattering amplitudes in gauge theory*, *JHEP* **07** (2004) 032 [[hep-th/0406085](#)];
 J.B. Wu and C.J. Zhu, *Mhv vertices and fermionic scattering amplitudes in gauge theory with quarks and gluinos*, *JHEP* **09** (2004) 063 [[hep-th/0406146](#)];
 D.A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, *Phys. Rev.* **D 71** (2005) 045007 [[hep-th/0406175](#)];
 X. Su and J.B. Wu, *Six-quark amplitudes from fermionic mhv vertices*, *Mod. Phys. Lett.* **A 20** (2005) 1065 [[hep-th/0409228](#)];
 T.G. Birthwright, E.W.N. Glover, V.V. Khoze and P. Marquard, *Multi-gluon collinear limits from MHV diagrams*, *JHEP* **05** (2005) 013 [[hep-ph/0503063](#)];
 T.G. Birthwright, E.W.N. Glover, V.V. Khoze and P. Marquard, *Collinear limits in QCD from MHV rules*, *JHEP* **07** (2005) 068 [[hep-ph/0505219](#)].
- [7] A. Brandhuber, B.J. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from MHV vertices*, *Nucl. Phys.* **B 706** (2005) 150 [[hep-th/0407214](#)];
 F. Cachazo, P. Svrcek and E. Witten, *Gauge theory amplitudes in twistor space and holomorphic anomaly*, *JHEP* **10** (2004) 077 [[hep-th/0409245](#)];
 M.X. Luo and C.K. Wen, *One-loop maximal helicity violating amplitudes in $N = 4$ super Yang-Mills theories*, *JHEP* **11** (2004) 004 [[hep-th/0410045](#)];
 I. Bena, Z. Bern, D.A. Kosower and R. Roiban, *Loops in twistor space*, *Phys. Rev.* **D 71** (2005) 106010 [[hep-th/0410054](#)];
 M.X. Luo and C.K. Wen, *Systematics of one-loop scattering amplitudes in $N = 4$ super Yang-Mills theories*, *Phys. Lett.* **B 609** (2005) 86 [[hep-th/0410118](#)];
 C. Quigley and M. Rozali, *One-loop mhv amplitudes in supersymmetric gauge theories*, *JHEP* **01** (2005) 053 [[hep-th/0410278](#)];
 J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A twistor approach to one-loop amplitudes in $N = 1$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 706** (2005) 100 [[hep-th/0410280](#)];
 J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *Non-supersymmetric loop amplitudes and MHV vertices*, *Nucl. Phys.* **B 712** (2005) 59 [[hep-th/0412108](#)].
- [8] L.J. Dixon, E.W.N. Glover and V.V. Khoze, *Mhv rules for Higgs plus multi-gluon amplitudes*, *JHEP* **12** (2004) 015 [[hep-th/0411092](#)];
 Z. Bern, D. Forde, D.A. Kosower and P. Mastrolia, *Twistor-inspired construction of electroweak vector boson currents*, *Phys. Rev.* **D 72** (2005) 025006 [[hep-ph/0412167](#)];

- S.D. Badger, E.W.N. Glover and V.V. Khoze, *Mhv rules for Higgs plus multi-parton amplitudes*, *JHEP* **03** (2005) 023 [[hep-th/0412275](#)].
- [9] N.E.J. Bjerrum-Bohr, D.C. Dunbar, H. Ita, W.B. Perkins and K. Risager, *MHV-vertices for gravity amplitudes*, *JHEP* **01** (2006) 009 [[hep-th/0509016](#)].
- [10] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [[hep-th/0412308](#)].
- [11] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [[hep-th/0501052](#)].
- [12] R. Roiban, M. Spradlin and A. Volovich, *Dissolving $N = 4$ loop amplitudes into QCD tree amplitudes*, *Phys. Rev. Lett.* **94** (2005) 102002 [[hep-th/0412265](#)].
- [13] M.X. Luo and C.K. Wen, *Recursion relations for tree amplitudes in super gauge theories*, *JHEP* **03** (2005) 004 [[hep-th/0501121](#)]; *Compact formulas for all tree amplitudes of six partons*, *Phys. Rev.* **D 71** (2005) 091501 [[hep-th/0502009](#)];
R. Britto, B. Feng, R. Roiban, M. Spradlin and A. Volovich, *All split helicity tree-level gluon amplitudes*, *Phys. Rev.* **D 71** (2005) 105017 [[hep-th/0503198](#)].
- [14] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A recursion relation for gravity amplitudes*, *Nucl. Phys.* **B 721** (2005) 98 [[hep-th/0502146](#)];
F. Cachazo and P. Svrcek, *Tree level recursion relations in general relativity*, [hep-th/0502160](#).
- [15] S.D. Badger, E.W.N. Glover, V.V. Khoze and P. Svrcek, *Recursion relations for gauge theory amplitudes with massive particles*, *JHEP* **07** (2005) 025 [[hep-th/0504159](#)].
- [16] S.D. Badger, E.W.N. Glover and V.V. Khoze, *Recursion relations for gauge theory amplitudes with massive vector bosons and fermions*, *JHEP* **01** (2006) 066 [[hep-th/0507161](#)].
- [17] D. Forde and D.A. Kosower, *All-multiplicity amplitudes with massive scalars*, [hep-th/0507292](#);
G. Rodrigo, *Multigluonic scattering amplitudes of heavy quarks*, *JHEP* **09** (2005) 079 [[hep-ph/0508138](#)].
- [18] Z. Bern, L.J. Dixon and D.A. Kosower, *On-shell recurrence relations for one-loop QCD amplitudes*, *Phys. Rev.* **D 71** (2005) 105013 [[hep-th/0501240](#)];
The last of the finite loop amplitudes in QCD, *Phys. Rev.* **D 72** (2005) 125003 [[hep-ph/0505055](#)];
Z. Bern, N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, *Recursive calculation of one-loop QCD integral coefficients*, *JHEP* **11** (2005) 027 [[hep-ph/0507019](#)].
- [19] K. Risager, *A direct proof of the CSW rules*, *JHEP* **12** (2005) 003 [[hep-th/0508206](#)].
- [20] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop N point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [[hep-ph/9403226](#)].
- [21] L.J. Dixon, *Calculating scattering amplitudes efficiently*, [hep-ph/9601359](#).
- [22] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [[hep-th/0412103](#)];
Z. Bern, V. Del Duca, L.J. Dixon and D.A. Kosower, *All non-maximally-helicity-violating one-loop seven-gluon amplitudes in $N = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 71** (2005) 045006 [[hep-th/0410224](#)];

- Z. Bern, L.J. Dixon and D.A. Kosower, *All next-to-maximally helicity-violating one-loop gluon amplitudes in $N = 4$ super-Yang-Mills theory*, *Phys. Rev. D* **72** (2005) 045014 [[hep-th/0412210](#)];
- S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *One-loop gluon scattering amplitudes in theories with $N < 4$ supersymmetries*, *Phys. Lett. B* **612** (2005) 75 [[hep-th/0502028](#)];
- R. Britto, E. Buchbinder, F. Cachazo and B. Feng, *One-loop amplitudes of gluons in SQCD*, *Phys. Rev. D* **72** (2005) 065012 [[hep-ph/0503132](#)].
- [23] A. Brandhuber, S. McNamara, B.J. Spence and G. Travaglini, *Loop amplitudes in pure Yang-Mills from generalised unitarity*, *JHEP* **10** (2005) 011 [[hep-th/0506068](#)].
- [24] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys. B* **435** (1995) 59 [[hep-ph/9409265](#)].
- [25] Z. Bern, L.J. Dixon and D.A. Kosower, *Bootstrapping multi-parton loop amplitudes in QCD*, [hep-ph/0507005](#).
- [26] Z. Bern, L.J. Dixon and D.A. Kosower, *One loop corrections to five gluon amplitudes*, *Phys. Rev. Lett.* **70** (1993) 2677 [[hep-ph/9302280](#)]; *One loop corrections to two quark three gluon amplitudes*, *Nucl. Phys. B* **437** (1995) 259 [[hep-ph/9409393](#)];
- Z. Kunszt, A. Signer and Z. Trócsányi, *One loop radiative corrections to the helicity amplitudes of QCD processes involving four quarks and one gluon*, *Phys. Lett. B* **336** (1994) 529 [[hep-ph/9405386](#)].
- [27] Callum Quigley and Moshe Rozali, in progress.
- [28] M.L. Mangano and S.J. Parke, *Multiparton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301 [[hep-th/0509223](#)].
- [29] Z. Bern and A.G. Morgan, *Massive loop amplitudes from unitarity*, *Nucl. Phys. B* **467** (1996) 479 [[hep-ph/9511336](#)].